Session 2 - Solutions to Exercises

> **Lemma 2.**
>
> *Let $\phi, \psi \in \mathcal{L}$ and let $\Gamma, \Delta \in \mathcal{P}(\mathcal{L})$. The semantic sets behave according to the following logical rules:*
>
> $$\begin{array}{ll} 1. & \phi \vdash \psi \iff [\phi] \subseteq [\psi] \\ 2. & \Gamma \subseteq \Delta \implies [\Delta] \subseteq [\Gamma] \\ 3. & \Gamma \subseteq \Delta \iff [\Delta] \subseteq [\Gamma] \quad \text{(if } \Delta \text{ is a belief set)} \\ 4. & [Cn(\Gamma \cup \Delta)] = [\Gamma] \cap [\Delta] \end{array}$$

Proof. Let $\phi, \psi \in \mathcal{L}$ and let $\Gamma, \Delta \in \mathcal{P}(\mathcal{L})$.

Part one. First, we prove the semantic equivalence for single sentences.

$$\begin{aligned} \phi \vdash \psi &\iff \text{for all } w \in W : w \models \phi \implies w \models \psi \\ &\iff \text{for all } w \in W : w \in [\phi] \implies w \in [\psi] && \text{by def. } w \models \phi \iff w \in [\phi] \\ &\iff [\phi] \subseteq [\psi] && \text{by def. of } \subseteq \end{aligned}$$

Next, we prove the same holds between a set of sentences Γ and a single sentence ϕ:

$$\begin{aligned} \Gamma \vdash \phi &\iff \text{for all } w \in W : (\text{if } w \models \gamma \text{ for all } \gamma \in \Gamma, \text{then } w \models \phi) \\ &\iff \text{for all } w \in W : (\text{if } w \in [\gamma] \text{ for all } \gamma \in \Gamma, \text{then } w \in [\phi]) && \text{by def. of } [\cdot] \\ &\iff \text{for all } w \in W : w \in \bigcap_{\gamma \in \Gamma} [\gamma] \implies w \in [\phi] && \text{by def. of intersection} \\ &\iff \text{for all } w \in W : w \in [\Gamma] \implies w \in [\phi] && \text{by def. of } [\Gamma] \\ &\iff [\Gamma] \subseteq [\phi] && \text{by def. of } \subseteq \end{aligned}$$

Part two. Suppose that $\Gamma \subseteq \Delta$, meaning that every sentence in Γ is also in Δ. We can visualize this as:

$$\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \ldots\} \subseteq \{\gamma_1, \gamma_2, \gamma_3, \ldots, \delta_1, \delta_2, \delta_3, \ldots\} = \Delta$$

(Note: we do not strictly need to assume the existence of extra δ sentences, but it helps visualize the inclusion).

Consider now any arbitrary world $w \in [\Delta]$. By definition, $[\Delta] = \bigcap_{\delta \in \Delta} [\delta]$. Since $w \in [\Delta]$, it follows that w satisfies all the sentences in Δ. Because $\Gamma \subseteq \Delta$, w must inherently satisfy all the sentences in Γ as well. That is, $w \in [\Gamma]$. Since w is an arbitrary world in $[\Delta]$, it follows that every world in $[\Delta]$ is also in $[\Gamma]$. Therefore:

$$[\Delta] \subseteq [\Gamma]$$

Part three. Suppose that Δ is a belief set, meaning $\Delta = Cn(\Delta)$. Let us prove the strict biconditional:

1. ($\implies$) Suppose $\Gamma \subseteq \Delta$. The conclusion $[\Delta] \subseteq [\Gamma]$ follows directly from the universal reasoning in Part two.
2. ($\impliedby$) Suppose now that $[\Delta] \subseteq [\Gamma]$. We must prove that $\Gamma \subseteq \Delta$. Let $\gamma \in \Gamma$. Since $[\Delta] \subseteq [\Gamma]$, every model of Δ is a model of γ, which means $\Delta \models \gamma$. By completeness (as shown in Part one), $\Delta \vdash \gamma$. This means $\gamma \in Cn(\Delta)$. Because we assumed Δ is a belief set, we know $Cn(\Delta) = \Delta$. Therefore, $\gamma \in \Delta$. Since γ is just an arbitrary sentence in Γ, it follows that all sentences in Γ are in Δ, hence $\Gamma \subseteq \Delta$.

Therefore, we conclude that, if Δ is a logically closed belief set, then for any set of sentences Γ:

$$\Gamma \subseteq \Delta \iff [\Delta] \subseteq [\Gamma]$$

Part four. We prove the equality by demonstrating mutual inclusion.

- ($\subseteq$) Let $w \in [Cn(\Gamma \cup \Delta)]$. By definition, w is a model of the deductive closure, meaning $w \models \psi$ for every formula $\psi \in Cn(\Gamma \cup \Delta)$. Since $\Gamma \cup \Delta \subseteq Cn(\Gamma \cup \Delta)$, it immediately follows that $w \models \psi$ for every $\psi \in \Gamma \cup \Delta$. This implies $w \in [\Gamma]$ and $w \in [\Delta]$, which yields $w \in [\Gamma] \cap [\Delta]$.
- ($\supseteq$) Let $w \in [\Gamma] \cap [\Delta]$. By definition, $w \in [\Gamma]$ and $w \in [\Delta]$, which means w is a model of all formulas in Γ and all formulas in Δ. Therefore, w is a model of their union: $w \models \Gamma \cup \Delta$. Now, let χ be an arbitrary formula such that $\chi \in Cn(\Gamma \cup \Delta)$, i.e. $\Gamma \cup \Delta \vdash \chi$. By definition of logical consequence, every world that satisfies $\Gamma \cup \Delta$ also satisfies χ. Since $w \models \Gamma \cup \Delta$, it follows that $w \models \chi$. Because χ was an arbitrary formula in the closure, w satisfies every formula in $Cn(\Gamma \cup \Delta)$. Thus, $w \in [Cn(\Gamma \cup \Delta)]$.

Since we have established inclusion in both directions, the equality $[Cn(\Gamma \cup \Delta)] = [\Gamma] \cap [\Delta]$ holds. □

Lemma 4.

Let $V, V_1, V_2 \subseteq W$ be any sets of worlds, and $\Gamma, \Delta \subseteq \mathcal{L}$ be any sets of sentences.

1. *$T(V)$ is a belief set, i.e. $T(V) = Cn(T(V))$.*
2. *$T([\Gamma]) = Cn(\Gamma)$. Consequently, $T([\Gamma]) = \Gamma \iff \Gamma$ is a belief set.*
3. *$V \subseteq [T(V)]$. Furthermore, the equality $[T(V)] = V$ holds for all $V \subseteq W$ if, and only if, Φ is finite.*
4. *If $V_1 \subseteq V_2$, then $T(V_2) \subseteq T(V_1)$.*

Proof. Let $V \subseteq W$ be any set of worlds, and $\Gamma \subseteq \mathcal{L}$ be any set of sentences.

Part one. Let us show that $T(V)$ is a logically closed belief set. Since any set is trivially a subset of its own deductive closure, we automatically have $T(V) \subseteq Cn(T(V))$.

To prove the reverse inclusion, suppose that $\phi \in Cn(T(V))$. We need to show that $\phi \in T(V)$.

1. Since $\phi \in Cn(T(V))$, we know by definition that $T(V) \vdash \phi$.
2. By our previous lemma RG FormEp - Session 2 > Lemma 2, $T(V) \vdash \phi$ implies that the models of the premise are a subset of the models of the conclusion: $[T(V)] \subseteq [\phi]$.
3. Furthermore, by definition, $T(V) = \{\psi \in \mathcal{L} : V \subseteq [\psi]\}$ and $[T(V)] = \{w \in W : w \models \psi \text{ for all } \psi \in T(V)\}$.
4. If $w \in V$, then $w \models \psi$ for all $\psi \in T(V)$, and hence $w \in [T(V)]$, which implies that $V \subseteq [T(V)]$.
5. By transitivity of subsets, $V \subseteq [T(V)] \subseteq [\phi]$, which means $V \subseteq [\phi]$.
6. Finally, by the definition of $T(V)$, any sentence whose truth-set contains V must be in $T(V)$. Therefore, $\phi \in T(V)$.

Since we have shown $Cn(T(V)) \subseteq T(V)$ and $T(V) \subseteq Cn(T(V))$, we conclude $T(V) = Cn(T(V))$.

Part two. (2a) Let us show that $T([\Gamma]) = Cn(\Gamma)$.

$$\begin{aligned} T([\Gamma]) &= \{\psi \in \mathcal{L} : [\Gamma] \subseteq [\psi]\} && \text{by def. of } T \\ &= \left\{\psi \in \mathcal{L} : \bigcap_{\gamma \in \Gamma} [\gamma] \subseteq [\psi]\right\} && \text{by def. of } [\Gamma] \\ &= \{\psi \in \mathcal{L} : \Gamma \vdash \psi\} && \text{by previous lemma} \\ &= Cn(\Gamma) && \text{by def. of } Cn \end{aligned}$$

(2b) Let us show that $T([\Gamma]) = \Gamma$ if, and only if, Γ is a belief set.

$$\begin{aligned} \Gamma \text{ is a belief set} &\iff \Gamma = Cn(\Gamma) && \text{by def. of belief set} \\ &\iff \Gamma = T([\Gamma]) && \text{by part (2a)} \end{aligned}$$

Part three. Let us prove that the truth-set function $[\cdot] : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(W)$ is the inverse of the theory function $T : \mathcal{P}(W) \to \mathcal{P}(\mathcal{L})$ if, and only if, the set of propositional variables Φ for our language $\mathcal{L}$ is finite.

($\Longleftarrow$) Suppose that Φ is finite. Let $V \in \mathcal{P}(W)$. We want to show that $[T(V)] = V$.

First, let me show that the inclusion $V \subseteq [T(V)]$ holds. By definition, $T(V) = \{\phi \in \mathcal{L} : V \subseteq [\phi]\}$. This simply means every formula in $T(V)$ is true in all worlds in V. Therefore, any world $w \in V$ must satisfy all formulas in $T(V)$, which by definition means $w \in [T(V)]$. Thus, $V \subseteq [T(V)]$.

Second, we show the reverse inclusion $[T(V)] \subseteq V$. Since Φ is finite, the set of all possible valuations W is also finite (specifically, $|W| = 2^{|\Phi|}$). Because W is finite, $V \subseteq W$ is also finite. For any world $w \in W$, we can construct a characteristic formula, or *atom*, that perfectly describes it:

$$\alpha(w) := \bigwedge_{p \in \Phi} l_w(p) \qquad \text{where } l_w(p) := \begin{cases} p & \text{if } w \models p \\ \neg p & \text{if } w \not\models p \end{cases}$$

By construction, $w' \models \alpha(w)$ if and only if $w' = w$. We can now construct a single formula ϕ_V that perfectly captures the set V by taking the disjunction of the atoms of all worlds in V:

$$\phi_V := \bigvee_{w \in V} \alpha(w)$$

(If V is empty, ϕ_V is simply $\bot$). It follows immediately that $[\phi_V] = V$.

Because $[\phi_V] = V$, we know that $T(V) = T([\phi_V])$. By Part two of our lemma, we also know that $T([\phi_V]) = Cn(\phi_V)$. Therefore:

$$T(V) = Cn(\phi_V)$$

Taking the truth-set of both sides, we get:

$$[T(V)] = [Cn(\phi_V)]$$

Also, note that the truth-set of a deductively closed theory generated by a single formula is just the truth-set of the formula itself. First, if $w \in [\phi_V]$, $w \in [Cn(\phi_V)]$ by the definition of $\vdash$. Second, if $w \in [Cn(\phi_V)]$, it follows that $w \models \phi'$ for all $\phi' \in Cn(\phi_V)$. Since $\phi_V \in Cn(\phi_V)$, we conclude $w \models \phi_V$, i.e. $w \in [\phi_V]$. Therefore, $[Cn(\phi_V)] = [\phi_V]$.

Since we established $[\phi_V] = V$ above, we conclude:

$$[T(V)] = V$$

Thus, $T(\cdot)$ and $[\cdot]$ are inverses of each other when Φ is finite.

($\Longrightarrow$) Suppose that Φ is not finite. Without loss of generality, let us assume that Φ is countably infinite, i.e., $|\Phi| = |\mathbb{N}| = \aleph_0$. To prove this direction, we must find at least one specific set of worlds $V \subseteq W$ such that $[T(V)] \neq V$. Since the inclusion $V \subseteq [T(V)]$ always holds (as established above), we simply need to find a set V where $V \subset [T(V)]$.

Let w^* be a specific world in W, and define our set as all worlds *except* w^*:

$$V = W \setminus \{w^*\}$$

We will prove that $[T(V)] = W \supset V$. To do so, since $V = W \setminus \{w^*\}$, we simply need to prove that $w^* \models \psi$ for all $\psi \in T(V)$.

Intuitively, the only single sentence that could distinguish the set $W \setminus \{w^*\}$ from the full space W would be the negation of the characteristic formula (the atom) of w^*. However, because Φ is infinite, this atom cannot exist as a well-formed formula (it would require an infinite conjunction). Consequently, the language is too "coarse" to surgically exclude just a single world. Ultimately, this means $T(V)$ collapses into the set of mere propositional tautologies.

To prove this formally, let ψ be an arbitrary formula in $T(V)$. By definition, this means ψ is true in every world in $V = W \setminus \{w^*\}$.

Given our definition of the language, any well-formed formula ψ has a finite length, meaning it can only contain a finite number of propositional variables. Let $Var(\psi) \subseteq \Phi$ be the set of variables that appear in ψ. Since Φ is infinite but $Var(\psi)$ is finite, there must be some variable $q \in \Phi$ that does *not* appear in ψ, i.e., $q \notin Var(\psi)$.

Now, let's consider a world w' that is identical to w^* in every way, except it flips the truth value of q. That is:

$$w'(p) = \begin{cases} w^*(p) & \text{If } p \neq q \\ 1 - w^*(p) & \text{If } p = q \end{cases}$$

Because w' and w^* differ on q, we know $w' \neq w^*$. Since $w' \in W$ but $w' \neq w^*$, it follows that w' must be inside our set V, for $V = W \setminus \{w^*\}$.

Since $\psi \in T(V)$ and $w' \in V$, it must be the case that $w' \models \psi$. However, ψ does not contain the variable q. Therefore, the truth value of ψ depends entirely on the variables in $Var(\psi)$, where w' and w^* perfectly agree. Hence, since $w' \models \psi$, it logically follows that $w^* \models \psi$.

Because ψ is an arbitrary formula in $T(V)$, this same reasoning applies to any other formula $\psi' \in T(V)$. We conclude that w^* satisfies every formula in $T(V)$. Therefore:

$$w^* \in [T(V)]$$

But by our initial definition, $w^* \notin V$. Therefore, $[T(V)]$ contains at least one world that is not in V. We have successfully proven that $V \subset [T(V)]$, and thus $[T(V)] \neq V$ when Φ is countably infinite. *A fortiori*, this result holds for uncountably infinite sets of propositional variables as well.

Part four. Suppose that $V_1 \subseteq V_2$, and let me prove that $T(V_2) \subseteq T(V_1)$. Consider an arbitrary $\phi \in T(V_2)$. By definition, $V_2 \subseteq [\phi]$. Since *ex hypothesi* $V_1 \subseteq V_2$, we have $V_1 \subseteq [\phi]$. Therefore, by definition $\phi \in T(V_1)$. Since ϕ is an arbitrary formula in $T(V_2)$, it follows that

$$T(V_2) \subseteq T(V_1)$$

⚠ Even if Φ is infinite, $[T(\{w\})] = \{w\}$

You might wonder why taking the theory of a *single* world doesn't suffer from the same "loss of information." The reason is that $T(\{w\})$ is an *infinite set* of formulas. It contains the literal $l_w(p)$ for every single $p \in \Phi$. Even though we cannot construct a single finite atom $\alpha(w)$ because the variables are infinite, the individual literals $l_w(\cdot)$ are all independently captured inside the set $T(\{w\})$.

Consequently, the truth-set $[T(\{w\})]$, which contains all worlds satisfying *every* $\phi \in T(\{w\})$, must only contain worlds that make $l_w(p)$ true for all $p \in \Phi$. The only world satisfying this infinite set of literals is w itself. Hence, $[T(\{w\})] = \{w\}$.